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Citation: **1798**, 020130 (2017); doi: 10.1063/1.4972722
View online: http://dx.doi.org/10.1063/1.4972722
View Table of Contents: http://aip.scitation.org/toc/apc/1798/1
Published by the American Institute of Physics

Some operational properties of the Laguerre transform

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Abstract. This paper is devoted to the study of some properties of the Laguerre transform. We define new properties of the Laguerre transform in a weighted L_2-space. Moreover, we present some results concerning the action of this integral transform over some class of polynomials.

INTRODUCTION

The Laguerre polynomials appear naturally in many branches of pure and applied mathematics and mathematical physics (see e.g. [1, 2, 3, 6]). Debnath [1] introduced the Laguerre transform and derived some of its properties. He also discussed the applications in study of heat conduction [3] and to the oscillations of a very long and heavy chain with variable tension [2]. Moreover, application of the integral Laguerre transforms for forward seismic modeling can be seen in [5].

The Laguerre transform of a function $f(x)$ is denoted by $\tilde{f}_\alpha(n)$ and defined by the integral

$$L\{f(x)\} = \widetilde{f_\alpha}(n) = \int_0^\infty e^{-x} x^\alpha L_n^\alpha(x) f(x) dx,\ n = 0, 1, 2, \dots \tag{0.1}$$

provided the integral exists in the sense of Lesbegue, where $L_n^\alpha(x)$ is a generalized Laguerre polynomial of degree n with order $\alpha > -1$, and satisfies the following differential equation

$$\frac{d}{dx}\left[e^{-x} x^{\alpha+1} \frac{d}{dx} L_n^\alpha(x)\right] + n e^{-x} x^\alpha L_n^\alpha(x) = 0. \tag{0.2}$$

The sequence of Laguerre polynomial $(L_n^\alpha(x))_{n=0}^\infty$ have the following property:

$$\int_0^\infty e^{-x} x^\alpha L_n^\alpha(x) L_m^\alpha(x) dx = \begin{pmatrix} n+\alpha \\ n \end{pmatrix} \Gamma(\alpha+1)\delta_{nm}, \tag{0.3}$$

where δ_{nm} is Kronecker function defined by

$$\delta_{nm} = \begin{cases} 1, & \text{if } n = m \\ 0, & \text{if } n \neq m \end{cases} \tag{0.4}$$

and $\Gamma(\alpha + 1) = \int_0^\infty x^\alpha e^{-x} dx$.

AIP Conf. Proc. 1798, 020130-1–020130-10; doi: 10.1063/1.4972722
Published by AIP Publishing. 978-0-7354-1464-8/$30.00

The inverse of the Laguerre transformation is then

$$f(x) = \sum_{n=0}^{\infty} (\delta_n)^{-1} \widetilde{f_\alpha}(n) L_n^\alpha(x) \ (0 < x < \infty),$$

where $\delta_n = \begin{pmatrix} n+\alpha \\ n \end{pmatrix} \Gamma(\alpha+1)$.

This paper is devoted to the study of the generalized Laguerre transform and some operational properties.

MAIN RESULTS

In this section we define new properties for the Laguerre transform in a weighted L_2-space. Moreover, we present some results concerning the action of this integral transform over some class of polynomial.

For $1 \leq p \leq \infty$ the space $L_{p,\alpha}$ is defined via the following formula

$$L_{p,\alpha} = \left\{ f : (0, \infty) \to \mathbb{R} : \ \int_0^\infty |f(x)|^p e^{-x} x^\alpha dx < \infty \right\}$$

with the norm

$$\|f\|_{L_{p,\alpha}} = \left(\int_0^\infty |f(x)|^p e^{-x} x^\alpha dx \right)^{1/p}$$

where the convention that

$$\|f\|_{L_{p,\alpha}} = \left(\int_0^\infty |f(x)|^p e^{-x} x^\alpha dx \right)^{1/p} = \text{ess sup}_{x \in \mathbb{R}} |f(x)|$$

if $p = \infty$.

Clearly, if f is an arbitrary polynomial then $f \in L_{p,\alpha}$. We also define

$$l_{p,\alpha} = \left\{ f = (f(n))_{n=0}^\infty : \ \sum_{n=0}^\infty |f(n)|^p \begin{pmatrix} n+\alpha \\ n \end{pmatrix} < \infty \right\}$$

with the norm

$$\|f\|_{l_{p,\alpha}} = \left(\sum_{n=0}^\infty |f(n)|^p \begin{pmatrix} n+\alpha \\ n \end{pmatrix} \right)^{1/p}.$$

We define the differential operator R via the formula

$$\begin{aligned} R[f(x)] &= e^x x^{-\alpha} \tfrac{d}{dx} [e^{-x} x^{\alpha+1} \tfrac{d}{dx} f(x)] \\ &= x f''(x) + (\alpha + 1 - x) f'(x) \end{aligned}$$

and then for $n = 0, 1, ...,$ we have

$$R^n[f(x)] = R[R^{n-1}[f(x)]].$$

Let $P(x)$ be a polynomial. The differential operator $P(R)$ is obtain from $P(x)$ by substituting $x \to R$. i.e., if $P(x) = \sum_{k=0}^n a_k x^k$ then

$$P(R)[f(x)] = \sum_{k=0}^n a_k R^k [f(x)].$$

Moreover, we denote by

$$\text{supp} \widetilde{f_\alpha} = \{ n \in \mathbb{Z}_+ : \ \widetilde{f_\alpha}(n) \neq 0 \}$$

the support of the Laguerre transform of f.

Now, we have the following theorem

Theorem 0.1 *Let $1 \le p < \infty$, $P(x)$ be a polynomial. Then for an arbitrary infinitely differentiable function $f \in L_{p,\alpha}$, there exist the following limit*

$$\lim_{m\to\infty} \|P^m(R)[f(x)]\|^{1/m}_{L_{p,\alpha}}$$

and

$$\lim_{m\to\infty} \|P^m(R)[f(x)]\|^{1/m}_{L_{p,\alpha}} = \sup\{|P(-n)| : \ n \in supp\widetilde{f}_\alpha\}.$$

Proof: From the definition of differential operator R, we get

$$\begin{aligned} L\{R[f(x)]\} &= \int_0^\infty e^{-x}x^\alpha L_n^\alpha(x)\, e^x x^{-\alpha} \tfrac{d}{dx}\left[e^{-x}x^{\alpha+1}\tfrac{df}{dx}\right]dx \\ &= \int_0^\infty L_n^\alpha(x)\tfrac{d}{dx}\left[e^{-x}x^{\alpha+1}\tfrac{df}{dx}\right]dx. \end{aligned}$$

Using the integral by part, we have

$$\begin{aligned} L\{R[f(x)]\} &= \left[e^{-x}x^{\alpha+1}L_n^\alpha(x)\frac{df}{dx}\right]_0^\infty - \int_0^\infty e^{-x}x^{\alpha+1}\frac{dL_n^\alpha(x)}{dx}\frac{df}{dx}dx \\ &= -\int_0^\infty e^{-x}x^{\alpha+1}\frac{dL_n^\alpha(x)}{dx}\frac{df}{dx}dx. \end{aligned} \tag{0.5}$$

We see that

$$\begin{aligned} -\int_0^\infty e^{-x}x^{\alpha+1}\frac{dL_n^\alpha(x)}{dx}\frac{df}{dx}dx &= \left[e^{-x}x^{\alpha+1}\frac{dL_n^\alpha(x)}{dx}f(x)\right]_0^\infty \\ &\quad + \int_0^\infty \frac{d}{dx}\left[e^{-x}x^{\alpha+1}\frac{dL_n^\alpha(x)}{dx}\right]f(x)\,dx \\ &= \int_0^\infty \frac{d}{dx}\left[e^{-x}x^{\alpha+1}\frac{dL_n^\alpha(x)}{dx}\right]f(x)\,dx. \end{aligned} \tag{0.6}$$

Then, from (0.2) it follows

$$-\int_0^\infty e^{-x}x^{\alpha+1}\frac{dL_n^\alpha(x)}{dx}\frac{df}{dx}dx = -n\int_0^\infty e^{-x}x^\alpha f(x)\,L_n^\alpha(x)\,dx.$$

Hence

$$L\{R[f(x)]\} = -n\widetilde{f}_\alpha(n). \tag{0.7}$$

Similarly, we get

$$L\{R^m[f(x)]\} = (-1)^m n^m \widetilde{f}_\alpha(n). \tag{0.8}$$

Hence, from this and the definition of $P^m(R)$ we obtain

$$L\{P^m(R)[f(x)]\} = P^m(-n)\widetilde{f}_\alpha(n). \tag{0.9}$$

Now, let consider σ an arbitrary number in supp$\widetilde{f}_\alpha$. Then

$$\int_0^\infty e^{-x}x^\alpha L_\sigma^\alpha(x)P^m(R)[f(x)]dx = R^m(-\sigma)\widetilde{f}_\alpha(\sigma). \tag{0.10}$$

Applying Hölder inequality, we obtain

$$\begin{aligned} \left|\int_0^\infty e^{-x}x^\alpha L_\sigma^\alpha(x)P^m(R)[f(x)]dx\right| &\le \left(\int_0^\infty e^{-x}x^\alpha |L_\sigma^\alpha(x)|^q dx\right)^{1/q} \\ &\quad \times\left(\int_0^\infty e^{-x}x^\alpha |P^m(R)[f(x)]|^p dx\right)^{1/p}. \end{aligned} \tag{0.11}$$

Therefore, it follows from (0.10) and (0.11) that

$$|P^m(-\sigma)\widetilde{f_\alpha}(\sigma)| \le \left(\int_0^\infty e^{-x}x^\alpha |L_\sigma^\alpha(x)|^q dx\right)^{1/q} \int_0^\infty e^{-x}x^\alpha |P^m(R)[f(x)]|^p dx. \tag{0.12}$$

That mean

$$|P^m(-\sigma)\widetilde{f_\alpha}(\sigma)| \le \left(\int_0^\infty e^{-x}x^\alpha |L_\sigma^\alpha(x)|^q dx\right)^{1/q} \|P^m(R)[f(x)]\|_{L_{p,\alpha}}. \tag{0.13}$$

As L_σ^α is a polynomial, we obtain $L_\sigma^\alpha \in L_{q,\alpha}$ and then

$$\left(\int_0^\infty e^{-x}x^\alpha |L_\sigma^\alpha(x)|^q dx\right)^{1/q} < \infty.$$

It follows from $\widetilde{f_\alpha}(\sigma) \neq 0$ and (0.13) that

$$|P(-\sigma)| \le \underline{\lim}_{m\to\infty} \|P^m(R)[f(x)]\|_{L_{p,\alpha}}^{1/m}. \tag{0.14}$$

Since (0.14) holds for all $\sigma \in \operatorname{supp}\widetilde{f_\alpha}$, we obtain

$$\sup\{|P(-\sigma)| : \ \ \sigma \in \operatorname{supp}\widetilde{f_\alpha}\} \le \underline{\lim}_{m\to\infty} \|P^m(R)[f(x)]\|_{L_{p,\alpha}}^{1/m}. \tag{0.15}$$

Now, we will prove that

$$\sup\{|P(-\sigma)| : \ \ \sigma \in \operatorname{supp}\widetilde{f_\alpha}\} \ge \overline{\lim}_{m\to\infty} \|P^m(R)[f(x)]\|_{L_{p,\alpha}}^{1/m}. \tag{0.16}$$

Indeed, if $\operatorname{supp}\widetilde{f_\alpha}$ is an unbounded set then

$$\sup\{|P(-\sigma)| : \ \sigma \in \operatorname{supp}\widetilde{f_\alpha}\} = \infty$$

and (0.16) holds. Now, we only need to prove (0.16) for the case that $\operatorname{supp}\widetilde{f_\alpha}$ is a bounded set, that mean, f is a polynomial and

$$M := \sup\{|n| : \ \ n \in \operatorname{supp}\widetilde{f_\alpha}\} < \infty.$$

Hence

$$P^m(R)[f(x)] = \sum_{n=-M}^{M} (\delta_n)^{-1} P^m(-n)\widetilde{f_\alpha}(n) L_n^\alpha(x) \ (0 < x < \infty)$$

and

$$\|P^m(R)[f(x)]\|_{L_{p,\alpha}} \le \textstyle\sum_{n=-M}^{M} (\delta_n)^{-1} P^m(-\sigma)\widetilde{f_\alpha}(n) \|L_n^\alpha\|_{L_{p,\alpha}} \tag{0.17}$$

$$\le [\sup\{|P(-\sigma)| : \ \ n \in \operatorname{supp}\widetilde{f_\alpha}\}]^m \textstyle\sum_{n=-M}^{M} (\delta_n)^{-1} \widetilde{f_\alpha}(n) \|L_n^\alpha\|_{L_{p,\alpha}} \tag{0.18}$$

which gives

$$\overline{\lim}_{m\to\infty} \|P^m(R)[f(x)]\|_{L_{p,\alpha}}^{1/m} \le \sup\{|P(-\sigma)| : \ \ \ \sigma \in \operatorname{supp}\widetilde{f_\alpha}\}.$$

Therefore, (0.16) have been proved. From (0.15) and (0.16), it is easy to see

$$\lim_{m\to\infty} \|P^m(R)[f(x)]\|_{L_{p,\alpha}}^{1/m} = \sup\{|P(-n)| : \ n \in \operatorname{supp}\widetilde{f_\alpha}\},$$

and then the proof is complete.

■

Let $P(x) = x$. Then, according Theorem 0.1 we obtain

Corollary 0.2 *Let $1 \leq p \leq \infty$, $P(x)$ be a polynomial. Then for an arbitrary arbitrary infinitely differentiable function $f \in L_{p,\alpha}$, there exist the following limit*

$$\lim_{m\to\infty} \|R^m[f(x)]\|_{L_{p,\alpha}}^{1/m}$$

and

$$\lim_{m\to\infty} \|R^m[f(x)]\|_{L_{p,\alpha}}^{1/m} = \deg(f).$$

Now, let consider an infinitely differentiable function $f : (0, \infty) \to \mathbb{R}$. Assume that we can define the sequence of function $(S^m[f(x)])_{m=1}^{\infty}$ satisfying the condition

$$R(S^m)[f(x)] = S^{m-1}[f(x)],\ \forall m \in \mathbb{N}.$$

We also see that, for an arbitrary polynomial f we get $\widetilde{f_\alpha}(0) = 0$ if and only if the sequence of function $(S^m[f(x)])_{m=1}^{\infty}$ is well defined and

$$S^m[f(x)] = \sum_{n=1}^{\infty} (\delta_n)^{-1} \left(\frac{-1}{n}\right)^m \widetilde{f_\alpha}(n)\, L_n^\alpha(x).$$

Next, we state the following theorem

Theorem 0.3 *Let consider $P(x)$ a polynomial and $1 \leq p \leq \infty$. Then for an arbitrary infinitely differentiable function $f \in L_{p,\alpha}$ satisfying $\widetilde{f_\alpha}(0) = 0$, we have*

$$\lim_{m\to\infty} \|P^m(S)[f(x)]\|_{L_{2,\alpha}}^{1/m} \geq \sup\left\{\left|P\left(\frac{-1}{n}\right)\right| :\ n \in \operatorname{supp}\widetilde{f_\alpha}\right\}.$$

Moreover, if f is a polynomial then

$$\lim_{m\to\infty} \|P^m(S)[f(x)]\|_{L_{2,\alpha}}^{1/m} = \sup\left\{\left|P\left(\frac{-1}{n}\right)\right| :\ n \in \operatorname{supp}\widetilde{f_\alpha}\right\}.$$

Proof: We have known that

$$L\{f(x)\} = L\{R(S)[f(x)]\} = nL\{S[f(x)]\}. \tag{0.19}$$

We deduce that

$$L\{S\,[f(x)]\} = \frac{-1}{n}\widetilde{f_\alpha}(n)\,. \tag{0.20}$$

Similarly, we get

$$L\{S^m\,[f(x)]\} = \left(\frac{-1}{n}\right)^m \widetilde{f_\alpha}(n)\,. \tag{0.21}$$

From this and the definition of $P^m(S)$, we obtain

$$L\{P^m(S)\,[f(x)]\} = P^m\left(\frac{-1}{n}\right)\widetilde{f_\alpha}(n)\,. \tag{0.22}$$

Now, we consider σ an arbitrary number in $\operatorname{supp}\widetilde{f_\alpha}$. Then

$$\int_0^\infty e^{-x}x^\alpha L_\sigma^\alpha(x)P^m(S)[f(x)]dx = R^m\left(\frac{-1}{\sigma}\right)\widetilde{f_\alpha}(\sigma). \tag{0.23}$$

Applying Hölder inequality we obtain

$$\begin{aligned}\left|\int_0^\infty e^{-x}x^\alpha L_\sigma^\alpha(x)P^m(S)[f(x)]dx\right| \quad &\leq \quad \left(\int_0^\infty e^{-x}x^\alpha |L_\sigma^\alpha(x)|^q dx\right)^{1/q} \\ &\quad\times\left(\int_0^\infty e^{-x}x^\alpha |P^m(S)[f(x)]|^p dx\right)^{1/p}.\end{aligned} \tag{0.24}$$

Therefore, it follows from (0.23) and (0.24) that

$$|P^m\left(\frac{-1}{\sigma}\right)\widetilde{f_\alpha}(\sigma)| \le \left(\int_0^\infty e^{-x}x^\alpha |L_\sigma^\alpha(x)|^q dx\right)^{1/q} \int_0^\infty e^{-x}x^\alpha |P^m(S)[f(x)]|^p dx.$$

That mean

$$|P^m\left(\frac{-1}{\sigma}\right)\widetilde{f_\alpha}(\sigma)| \le \left(\int_0^\infty e^{-x}x^\alpha |L_\sigma^\alpha(x)|^q dx\right)^{1/q} \|P^m(S)[f(x)]\|_{L_{p,\alpha}}. \tag{0.25}$$

As L_σ^α is a polynomial, we obtain $L_\sigma^\alpha \in L_{q,\alpha}$ and then

$$\left(\int_0^\infty e^{-x}x^\alpha |L_\sigma^\alpha(x)|^q dx\right)^{1/q} < \infty.$$

Then it follows from $\widetilde{f_\alpha}(\sigma) \neq 0$ and (0.25) that

$$|P\left(\frac{-1}{\sigma}\right)| \le \underline{\lim}_{m\to\infty} \|P^m(S)[f(x)]\|_{L_{p,\alpha}}^{1/m}. \tag{0.26}$$

Since (0.26) holds for all $\sigma \in \mathrm{supp}\widetilde{f_\alpha}$, we obtain

$$\sup\{|P\left(\frac{-1}{\sigma}\right)| : \ \sigma \in \mathrm{supp}\widetilde{f_\alpha}\} \le \underline{\lim}_{m\to\infty} \|P^m(S)[f(x)]\|_{L_{p,\alpha}}^{1/m}. \tag{0.27}$$

Now, we will prove

$$\sup\{|P\left(\frac{-1}{\sigma}\right)| : \ \sigma \in \mathrm{supp}\widetilde{f_\alpha}\} \ge \overline{\lim}_{m\to\infty} \|P^m(S)[f(x)]\|_{L_{p,\alpha}}^{1/m} \tag{0.28}$$

in the case where f is a polynomial.

Indeed, put $M := \deg(f)$. Hence,

$$P^m(S)[f(x)] = \sum_{n=-M}^{M} (\delta_n)^{-1} P^m\left(\frac{-1}{n}\right)\widetilde{f_\alpha}(n) L_n^\alpha(x) \ (0 < x < \infty)$$

and

$$\begin{aligned} \|P^m(S)[f(x)]\|_{L_{p,\alpha}} &\le \textstyle\sum_{n=-M}^{M} (\delta_n)^{-1} P^m\left(\frac{-1}{\sigma}\right)\widetilde{f_\alpha}(n)\|L_n^\alpha\|_{L_{p,\alpha}} \\ &\le [\sup\{|P(\tfrac{-1}{\sigma})| : \quad n \in \mathrm{supp}\widetilde{f_\alpha}\}]^m \textstyle\sum_{n=-M}^{M} (\delta_n)^{-1}\widetilde{f_\alpha}(n)\|L_n^\alpha\|_{L_{p,\alpha}} \end{aligned}$$

which gives

$$\overline{\lim}_{m\to\infty} \|P^m(S)[f(x)]\|_{L_{p,\alpha}}^{1/m} \le \sup\{|P\left(\frac{-1}{\sigma}\right)| : \quad \sigma \in \mathrm{supp}\widetilde{f_\alpha}\}.$$

That mean (0.28) have been proved. From (0.27) and (0.28), it is easy to see

$$\lim_{m\to\infty} \|P^m(S)[f(x)]\|_{L_{p,\alpha}}^{1/m} = \sup\{|P\left(\frac{-1}{n}\right)| : \ n \in \mathrm{supp}\widetilde{f_\alpha}\}$$

and the proof is complete.

■

Let $P(x) = x$, then we have the following result

Corollary 0.4 *Let $1 \leq p \leq \infty$ and let f an arbitrary polynomial satisfying $\widetilde{f_\alpha}(0) = 0$. Then, there exist the following limit*

$$\lim_{m\to\infty} \|S^m[f(x)]\|_{L_{p,\alpha}}^{1/m}$$

and

$$\lim_{m\to\infty} \|S^m[f(x)]\|_{L_{p,\alpha}}^{1/m} = \frac{1}{\deg_1(f)},$$

where $\deg_1(f) = \inf\{n \in \mathbb{Z}_+ : \widetilde{f_\alpha}(n) \neq 0\}$.

We denote by T_m the set of the algebra polynomials of degree $\leq m$, and the error of approximation $E_m(f)$ of f by elements from T_m is

$$E_m(f) = \inf_{P\in T_m} \|f - P\|_{L_{2,\alpha}}.$$

Take it in mind, we have the following theorem

Theorem 0.5 *Assume that $f(x) \in L_{2,\alpha}$. Then $L\{f(x)\} \in l_{2,\alpha}$ and*

$$\|f\|_{L_{2,\alpha}} = \sqrt{\Gamma(\alpha+1)}\|L\{f(x)\}\|_{l_{2,\alpha}}.$$

Moreover,

$$E_m(f) = \|f\|_{L_{2,\alpha}} - \sum_{n=0}^{m} (\widetilde{f_\alpha}(n))^2 \Gamma(\alpha+1) \binom{n+\alpha}{n}.$$

Proof: We see that

$$\begin{aligned} \| \quad f(x) - \textstyle\sum_{n=0}^m (\delta_n)^{-1} c(n) L_n^\alpha(x)\|_{L_{2,\alpha}} &= \int_0^\infty e^{-x}x^\alpha \Big(f(x) - \textstyle\sum_{n=0}^m (\delta_n)^{-1} c(n) L_n^\alpha(x)\Big)^2 dx & (0.29)\\ &= \int_0^\infty e^{-x}x^\alpha (f(x))^2 dx - 2\int_0^\infty e^{-x}x^\alpha f(x) \textstyle\sum_{n=0}^m (\delta_n)^{-1} c(n) L_n^\alpha(x) dx & (0.30)\\ &\quad + \int_0^\infty e^{-x}x^\alpha \left(\textstyle\sum_{n=0}^m (\delta_n)^{-1} c(n) L_n^\alpha(x)\right)^2 dx. & (0.31)\end{aligned}$$

Note that

$$\begin{aligned} \int_0^\infty e^{-x}x^\alpha f(x) \sum_{n=0}^{m} (\delta_n)^{-1} c(n) L_n^\alpha(x) dx &= \textstyle\sum_{n=0}^m \int_0^\infty e^{-x}x^\alpha f(x) (\delta_n)^{-1} c(n) L_n^\alpha(x) dx & (0.32)\\ &= \textstyle\sum_{n=0}^m (\delta_n)^{-1} c(n) \widetilde{f_\alpha}(n). & (0.33)\end{aligned}$$

Using (0.3) we obtain

$$\begin{aligned} \int_0^\infty e^{-x}x^\alpha \Big(\sum_{n=0}^{m} (\delta_n)^{-1} c(n) L_n^\alpha(x)\Big)^2 dx &= \textstyle\sum_{1\leq i,j\leq m} \int_0^\infty e^{-x}x^\alpha \Big((\delta_n)^{-2}(c(n))^2 L_i^\alpha(x) L_j^\alpha(x)\Big) dx \\ &= \textstyle\sum_{1\leq i=j\leq m} \int_0^\infty e^{-x}x^\alpha \Big((\delta_n)^{-2}(c(n))^2 L_i^\alpha(x) L_j^\alpha(x)\Big) dx \\ &= \textstyle\sum_{n=0}^m (\delta_n)^{-1} (c(n))^2. & (0.34)\end{aligned}$$

Using (0.29), (0.32) and (0.34) we get

$$\begin{aligned} \|f(x) - \sum_{n=0}^{m} (\delta_n)^{-1} c(n) L_n^\alpha(x)\|_{L_{2,\alpha}} &= \|f\|_{L_{2,\alpha}}^2 - 2\textstyle\sum_{n=0}^m (\delta_n)^{-1} c(n) \widetilde{f_\alpha}(n) + \sum_{n=0}^m (\delta_n)^{-1} (c(n))^2 \\ &= \|f\|_{L_{2,\alpha}}^2 - \textstyle\sum_{n=0}^m (\delta_n)^{-1} (\widetilde{f_\alpha}(n))^2 + \sum_{n=0}^m (\delta_n)^{-1} (\widetilde{f_\alpha}(n) - c(n))^2 & (0.35)\end{aligned}$$

which gives

$$\inf_{(c_0,\dots,c_m)\in\mathbb{R}^m} \|f(x) - \sum_{n=0}^{m} (\delta_n)^{-1} c(n) L_n^\alpha(x)\|_{L_{2,\alpha}} = \|f\|_{L_{2,\alpha}}^2 - \sum_{n=0}^{m} (\delta_n)^{-1} (\widetilde{f_\alpha}(n))^2.$$

Then, it follows from $\sum_{n=0}^{m}(\delta_n)^{-1}c(n)L_n^{\alpha}(x)$ which is the algebra polynomial of degree m

$$E_m(f) = \|f\|_{L_{2,\alpha}}^2 - \sum_{n=0}^{m}(\delta_n)^{-1}(\widetilde{f_\alpha}(n))^2 = \|f\|_{L_{2,\alpha}} - \sum_{n=0}^{m}(\widetilde{f_\alpha}(n))^2\Gamma(\alpha+1)\begin{pmatrix} n+\alpha \\ n \end{pmatrix}.$$

Let $c(n) = \widetilde{f_\alpha}(n)$ for all $n = 0, 1, \ldots, m$, we obtain

$$\|f(x) - \sum_{n=0}^{m}(\delta_n)^{-1}\widetilde{f_\alpha}(n)L_n^{\alpha}(x)\|_{L_{2,\alpha}} = \|f\|_{L_{2,\alpha}}^2 - \sum_{n=0}^{m}(\delta_n)^{-1}(\widetilde{f_\alpha}(n))^2.$$

Therefore, since $\lim\limits_{m\to\infty}\|f(x) - \sum\limits_{n=0}^{m}(\delta_n)^{-1}\widetilde{f_\alpha}(n)L_n^{\alpha}(x)\|_{L_{2,\alpha}} = 0$, we have $L\{f(x)\} \in l_{2,\alpha}$ and

$$\|f\|_{L_{2,\alpha}}^2 = \sum_{n=0}^{m}(\delta_n)^{-1}(\widetilde{f_\alpha}(n))^2,$$

and then

$$\sqrt{\Gamma(\alpha+1)}\|L\{f(x)\}\|_{l_{2,\alpha}} = \|f\|_{L_{2,\alpha}}.$$

The proof is complete.

■

We are now able to present the following result

Theorem 0.6 *Let $P(x)$ be the polynomial. Then for an arbitrary polynomial f, we always have*

$$\|P^m(R)[f(x)]\|_{L_{2,\alpha}} \leq \sup\{|P(-n)| : \ n \in supp\widetilde{f_\alpha}\}\|P^{m-1}(R)[f(x)]\|_{L_{2,\alpha}}, \tag{0.36}$$

$$\|P^m(R)[f(x)]\|_{L_{2,\alpha}} \geq \inf\left\{|P(-n)| : \ n \in supp\widetilde{f_\alpha}\right\}\|P^{m-1}(R)[f(x)]\|_{L_{2,\alpha}} \tag{0.37}$$

and

$$\lim_{m\to\infty}\|P^m(R)[f(x)]\|_{L_{2,\alpha}}^{1/m} = \deg(f). \tag{0.38}$$

Proof: We have known that

$$L\left\{P^m(R)\left[f(x)\right]\right\} = P^m(-n)\,\widetilde{f_\alpha}(n).$$

Then it follows from Theorem 0.5 that

$$\begin{aligned} \|P^m(R)[f(x)]\|_{L_{2,\alpha}} &= \sqrt{\Gamma(\alpha+1)}\|L\left\{P^m(R)\left[f(x)\right]\right\}\|_{l_{2,\alpha}} \\ &= \sqrt{\Gamma(\alpha+1)}\left(\textstyle\sum_{n=0}^{\infty}\left(P^m(-n)\widetilde{f_\alpha}(n)\right)^2\begin{pmatrix} n+\alpha \\ n \end{pmatrix}\right)^{1/2}. \end{aligned}$$

Hence

$$\begin{aligned} \|P^m(R)\left[f(x)\right]\|_{L_{2,\alpha}} &= \sqrt{\Gamma(\alpha+1)}\left(\sum_{n\in\mathrm{supp}\widetilde{f_\alpha}}\left(P^m(-n)\widetilde{f_\alpha}(n)\right)^2\begin{pmatrix} n+\alpha \\ n \end{pmatrix}\right)^{1/2} \\ &\leq \sqrt{\Gamma(\alpha+1)}\sup\left\{|P(-n)| : \ n \in \mathrm{supp}\widetilde{f_\alpha}\right\} \\ &\quad\times\left(\sum_{n\in\mathrm{supp}\widetilde{f_\alpha}}\left(P^{m-1}(-n)\widetilde{f_\alpha}(n)\right)^2\begin{pmatrix} n+\alpha \\ n \end{pmatrix}\right)^{1/2} \\ &= \sup\left\{|P(-n)| : \ n \in \mathrm{supp}\widetilde{f_\alpha}\right\}\|P^{m-1}(R)\left[f(x)\right]\|_{L_{2,\alpha}} \end{aligned}$$

and (0.36) just been proved.

Now, we will prove (0.37). Indeed, we have

$$\|P^m(R)[f(x)]\|_{L_{2,\alpha}} = \sqrt{\Gamma(\alpha+1)}\left(\sum_{n=0}^{\infty}(P^m(-n)\widetilde{f_\alpha}(n))^2\begin{pmatrix} n+\alpha \\ n \end{pmatrix}\right)^{1/2}$$

which gives

$$\|P^m(R)[f(x)]\|_{L_{2,\alpha}} = \sqrt{\Gamma(\alpha+1)}\left(\sum_{n\in\operatorname{supp}\widetilde{f_\alpha}}(P^m(-n)\widetilde{f_\alpha}(n))^2\begin{pmatrix} n+\alpha \\ n \end{pmatrix}\right)^{1/2}.$$

It follows that

$$\begin{aligned}\|P^m(R)[f(x)]\|_{L_{2,\alpha}} &\geq \sqrt{\Gamma(\alpha+1)}\inf\{|P(-n)| : n\in\operatorname{supp}\widetilde{f_\alpha}\}\\ &\quad\times\left(\sum_{n\in\operatorname{supp}\widetilde{f_\alpha}}(P^{m-1}(-n)\widetilde{f_\alpha}(n))^2\begin{pmatrix} n+\alpha \\ n \end{pmatrix}\right)^{1/2}\\ &= \inf\{|P(-n)| : n\in\operatorname{supp}\widetilde{f_\alpha}\}\|P^{m-1}(R)[f(x)]\|_{L_{2,\alpha}},\end{aligned}$$

and (0.37) just been proved. Note that, we check (0.37) from Theorem 0.1. The proof is complete.

■

Since for all a polynomial f and $n > \deg(f)$ then $\widetilde{f_\alpha}(n) = 0$, we obtain $\operatorname{supp}\widetilde{f_\alpha} \subset \{0, 1, \ldots \deg(f)\}$. Then, by applying Theorem 0.6 we have the following corollary

Corollary 0.7 *Let $P(x)$ be the polynomial. Then for an arbitrary polynomial f, we always have*

$$\|P^m(R)[f(x)]\|_{L_{2,\alpha}} \leq \sup\{|P(-n)| : 0\leq n\leq \deg(f)\}\|P^{m-1}(R)[f(x)]\|_{L_{2,\alpha}},$$

$$\|P^m(R)[f(x)]\|_{L_{2,\alpha}} \geq \inf\{|P(-n)| : 0\leq n\leq \deg(f)\}\|P^{m-1}(R)[f(x)]\|_{L_{2,\alpha}}$$

and

$$\lim_{m\to\infty}\|P^m(R)[f(x)]\|_{L_{2,\alpha}}^{1/m} = \deg(f).$$

Let $P(x) = x$, then it follows from Theorem 0.6

Corollary 0.8 *Let $P(x)$ be the polynomial. Then for an arbitrary polynomial f, we always have*

$$\|R^m[f(x)]\|_{L_{2,\alpha}} \leq \deg(f)\|P^{m-1}(R)[f(x)]\|_{L_{2,\alpha}},$$

$$\|R^m[f(x)]\|_{L_{2,\alpha}} \geq \deg_1(f)\|P^{m-1}(R)[f(x)]\|_{L_{2,\alpha}},$$

where $\deg_1(f) = \inf\{n\in\mathbb{Z}_+ : \widetilde{f_\alpha}(n)\neq 0\}$ *and*

$$\lim_{m\to\infty}\|R^m[f(x)]\|_{L_{2,\alpha}}^{1/m} = \deg(f).$$

Remark. Since f is the polynomial with degree d, $f(x)$ can express via

$$f(x) = \sum_{n=0}^{d} a_n L_\alpha^n(x).$$

Then

$$\{n\in\mathbb{Z}_+ : \widetilde{f_\alpha}(n)\neq 0\} = \{n\in\mathbb{N} : a_n\neq 0\}$$

and

$$C = \inf\{|P(-n)| : n\in\mathbb{Z}_+, \widetilde{f_\alpha}(n)\neq 0\} = \inf\{|P(-n)| : n\in\mathbb{Z}_+, a_n\neq 0\}.$$

ACKNOWLEDGMENTS

The work of the first author was supported by Portuguese funds through the CIDMA - Center for Research and Development in Mathematics and Applications, and the Portuguese Foundation for Science and Technology ("FCT–Fundação para a Ciência e a Tecnologia"), within project UID/MAT/ 0416/2013. The work of V.N.Huy was supported by Vietnam National University under grant number QG.16.08.

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